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Automated reasoning in elementary geometry: towards inquiry learning

Beatrix Hauer[a], Zoltán Kovács[a1], Tomás Recio[b1], M. Pilar Vélez[c1]

[a] The Private University College of Education of the Diocese of Linz, Austria
[b] University of Cantabria, Santander, Spain, [c] University Antonio de Nebrija, Madrid, Spain
zoltan@geogebra.org

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We claim that the inquiry-based learning approach to geometry can be improved by considering a recently implemented algorithm for the automatic conjecturing and proving of elementary geometry statements. The new method we describe relies on some Automated Reasoning Tools commands, now available in the dynamic geometry software GeoGebra, that provide with mathematically rigorous answers to any query posed by a user about the truth or falsity of any geometric statement and that, if the conjectured statement is wrong, present further hypotheses that should be considered for the proposition to become true. We argue, by providing examples of mathematical investigations that may be better approached by the students by using our method during the exploration process, how these tools may be helpful for supporting inquiry-based learning at various education levels. Some of the potential implications of our proposal (such as providing to automated reasoning programs, for the first time, the possibility of a real impact in education, or the possibility of automatizing the development of automated mentors, etc.) are briefly stated. It is yet ongoing work to develop a systematic approach to confirm the benefits of the proposed method.

KEYWORDS: mathematics education, GeoGebra, elementary geometry, conjecture, proof

1. Introduction

Educators' discussions on the potential role of software programs dealing with automatic theorem proving already appeared in papers published 30 years ago. The ICMI Study “School Mathematics in the 1990's” (Howson & Wilson, 1986) or the paper by P. Davis (1995), with a section that refers to the “transfiguration” power of computer-based proofs of geometry statements are clear examples of considerations about the future.

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The mathematical and computational background in the late 80s was already mature enough to allow implementing effective algorithms on automated reasoning in planar geometry. One of the remarkable first examples of successful automated experiments appear in the revolutionary book Chou (1988) that contains mechanical proofs of 512 geometry theorems, both including well known results and new ones. Later on, different several fruitful implementations have been made public such as the software programs *Discover* (Botana et al., 2002), *GEOTHER* (Wang, 2004), *GeoProof* (Narboux, 2007), *Geometry Expert* (Ye et al., 2011) and *GeoGebra Automated Reasoning Tools* ("GG-ART", Botana et al., 2015, Abánades et al., 2016), among others.

On the other hand, as described in the recent review by Tessier-Baillargeon et al. (2017), several works have already addressed the development of intelligent tutorial systems to help students with proofs in geometry, such as *GRAMY* (Matsuda & Vanlehn, 2004), *GeoGebraTutor* (Tessier-Baillargeon el al., 2014) or *QED-Tutrix* (Leduc, 2016). It must be remarked that these computer-based, intelligent tutors do not rely on automated theorem proving programs: they rather emphasize the automated mentoring aspects on some concrete geometric situations through solving strategies previously stored in the program memory.

Yet both tools, automated *tutorial* and automated *reasoning* in geometry, seem to have had little impact till now in the classroom. Thus, a recent survey by Sinclair et al. (2016), including a full section on the role of technologies on geometry education and another one on "Advances in the understanding of the teaching and learning of the proving process", do not include any reference to them.

In this context, our paper aims to illustrate the prospective classroom use of devices implementing automated reasoning in geometry, namely, by demonstrating, through the detailed description of some typical examples, the possibilities of GG-ART in an inquiry-based approach to mathematics education. This claim can be understood as the main thesis of this paper. As stated in Kovács et al., 2017, our proposal considers that "… GeoGebra ART is not merely a black box that produces effects or reactions to actions determined by a waiting user. In fact, just as the ancients were questioning an oracle to predict what would happen in a given context, the user employs an ART as a guiding stick in the geometric environment."

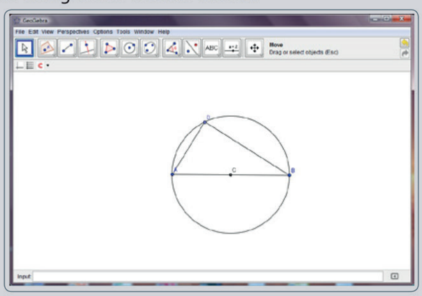
As a main difference with the case of intelligent tutors, in our work the automatism is not on the tutorial side, but on the availability for the student, through GG-ART, of some "omniscient" mathematical mentor, able to correctly answer his/her questions. Roughly speaking, the *Automated Reasoning Tools*, embedded in the dynamic geometry software GeoGebra, are able to output mathematically rigorous answers to any query posed by a user about the truth or falsity of any geometric statement. Moreover, if the conjectured statement turns out to be wrong, GG-ART present further hypotheses that should be considered for the proposition to become true.

In this context, the key scenario we are regarding for the application of GG-ART is neither that of helping the student in some straightforward proving tasks (requiring just a yes/no answer), nor addressing "pseudo-experimental activities" (see Artigue, 2012, Figure 1), leading students "step by step along a worksheet". Rather, in our paper we are considering authentic inquiry-based approaches, requiring the handling of geometric situations through the formulation of different conjectures that could be checked by obtaining reliable and mathematically rigorous information from the computer, by means of automated proofs being internally performed by the machine.

FIGURE 1. *Example of a pseudo-experimental activity provided by Artigue (2012)*

Box 3. Triangle and circle – the right angle theorem

Students work with an interactive geometry application such as the one captured in the screen shot below. D is a mobile point on the perimeter of a circle of diameter [AB]. Students are asked to make conjectures about the values of the angles of the triangle ABD when D moves.



The aim is that students observe that the angle in D is always 90°, and thus conjecture an important theorem in elementary geometry. But in this learning situation everything is given, and thus, despite appearances, there is no real place for mathematical inquiry. Note nevertheless that a small change would be sufficient to create a substantial difference. For instance, students could be given the segment [AB] and a point C that would be mobile in the plane, and asked to delimitate the region of the plane where point C could stand so that the angle in C of the triangle ABC is acute.

2. GeoGebra Automated Reasoning Tools

GG-ART are a collection of GeoGebra features and commands that allow to conjecture, discover, adjust and prove geometric statements in a dynamic geometric construction. Kovács et al. (2018) includes a complete tutorial illustrated with examples of these features.

To begin with GG-ART we have to draw a geometric construction in GeoGebra. Then we will exhibit the many possibilities that GeoGebra offers to enhance investigating and conjecturing about geometric properties of our construction. Say: investigating visually; using the Relation tool to compare objects and to obtain relations; or using the Locus tool to learn about the trace of a point subject to some constraints. These methods are usually well known by the GeoGebra community and well documented at the GeoGebra Materials website (https://www.geogebra.org/materials). But these methods are mostly numerical, i.e. not mathematically rigorous, they only work on the specific construction with concrete coordinates, so they do not allow to deal with general statements.

GG-ART bring to GeoGebra new capabilities for automatic reasoning in Euclidean plane geometry in an exact way, by using symbolic computations behind the concrete construction:

- The Relation tool and command can be now used to re-compute the results symbolically. When using the Relation tool, the user points on two objects (three or four objects can be also selected) and gets a message box. This message box is shown with one or more true numerical statements on the objects, there may be a button “More...” shown if there is symbolic support for the given statement. When clicking “More...”, shortly the numerical statement will be updated to a more general symbolic one, stating or denying the validity of the relation for arbitrary instances of the given construction. We can use alternatively the following command: **Relation[<Object>, <Object>]**.
- The LocusEquation command refines the result of the Locus command by displaying the algebraic equation of the graphical output, allowing to investigate and conjecture statements. The command has two forms:
  (1) Its first, *explicit* form returns the algebraic equation of the output of the Locus command if an algebraic translation of the geometric setup is possible. That is, the mover point M conducts some dependent objects of it, including the tracer point T. The syntax of the command is: **LocusEquation[<Tracer Point>, <Mover Point>]**.
  (2) Its *second*, implicit form starts by considering some construction steps and an input point P, either as a free point, or on a path in the construction. Then, the user claims a Boolean condition that holds on some objects of the construction. The task is to determine an equation $f(x,y)=0$ such that for all points $Q=(x,y)$ of its geometric representation, if $P=Q$, then the given condition holds. Here $f(x,y)=0$ is called locus equation, and its graphical representation is the locus. The syntax of the command is: **LocusEquation[<Boolean Expression>, <Point>]**.

There are also some other tools available in GG-ART for advanced uses, including envelope equation computations and direct proof of geometry statements. They are fully documented in Kovács et al. (2018).

3. Examples

Thales' circle theorem

To address Artigue's proposal on keeping the information secret that the locus of the solutions is related to a circle we demonstrate how Thales' theorem may be taught via inquiry by using GG-ART. We also refer to Kovács and Schiffler (2017) that sketches up a possible approach.

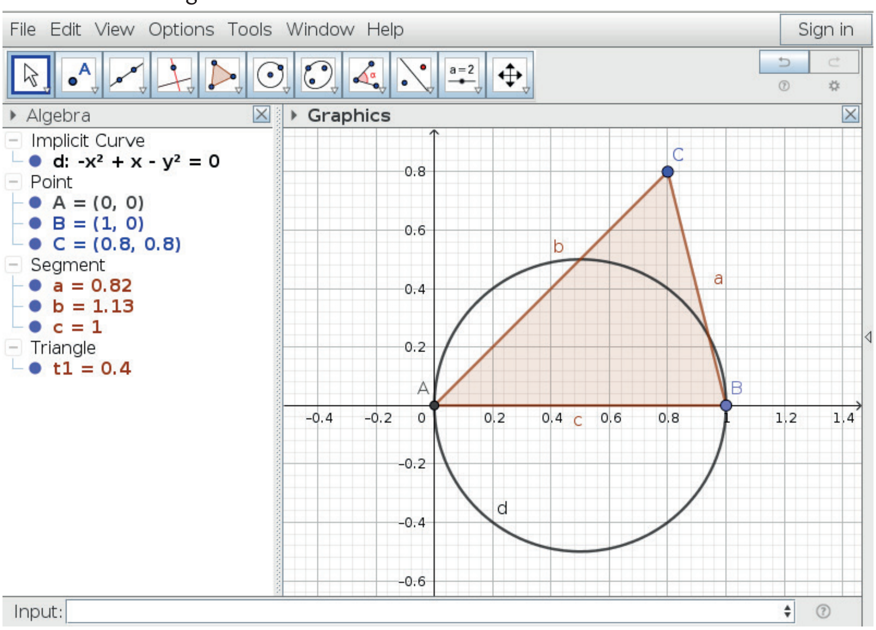
An inquiry-based approach may consist of the following steps:

1. It is crucial that the students need some mathematical knowledge about the *algebraic equations* of geometric objects including lines, circles, and also points (in the form of circles that have 0 radius).
2. Introduction of technical prerequisites for the students, including
   a) *basic GeoGebra tools,*
   b) and *LocusEquation command in its implicit (second) form*.

For the latter, here a key expectation is that the teacher should select introductory examples that enlighten the purpose of the LocusEquation command, but the theorem itself will be kept secret. Some possible introductory examples can be found in Lambert (2017).

3. *Collecting some experimental data* by using dynamic geometry based on experiments from step 2a. The segment *AB* is given and the students have to find possible locations of vertex *C* in order to have the triangle *ABC* right. A possible prototype of this task can be found at https://www.geogebra.org/m/scequnqq as a GeoGebraBook. We emphasize here that this experiment is purely numerical, that is no automated reasoning is used at this point yet.

FIGURE 2. *Discovering Thales' circle theorem in GeoGebra*



4. Based on experiments from step 2b the students should formalize a GeoGebra command, namely **LocusEquation(a⊥b,C)** to get *a proper answer for a par-*

ticular position of AB (see Figure 2). Visually it should be clear that the obtained curve is a circle, but for a justified answer mathematical knowledge from step 1 is also required. (In this particular position the equation should be rewritten to $(x-1/2)^2+y^2=(1/2)^2$.) The equation of the circle is computed symbolically in GeoGebra: this step already uses automated reasoning.

5. By *dragging* the end points of *AB* the locus curve will change dynamically. As a visual experiment it should be clear again that the curve always remains a circle. By investigating *several positions* algebraically also, by using step 1 again consecutively, there will be several examples where the conjecture holds. Here we emphasize that the speed of the symbolic and numerical algorithms used during automated reasoning are crucial in this experiment to get enough examples to get convinced.
6. After having a strong conjecture, the students can get a closer look on the theorem by *constructing* a circle with diameter *AB* and check it, by using the Relation tool, if *a* and *b* are indeed perpendicular. For this step, again, symbolic computations are used under the hood, but they are not shown to avoid the very technical details of the algebraic proof.
7. Eventually, if there is enough interest and time for that in the classroom, a *classic proof* can also be performed to get a rigorous argumentation. Here we refer to the well known classic methods.
8. Another inquiry can be initiated after concluding the truth of Thales' statement. Namely, a generalization of it: What kind of statement(s) can be obtained if the angle between *a* and *b* is not necessarily perpendicular? A possible solution of this task is sketched up at https://www.geogebra.org/m/afpvF72v#material/RnKahyCS.

Further examples from the curriculum

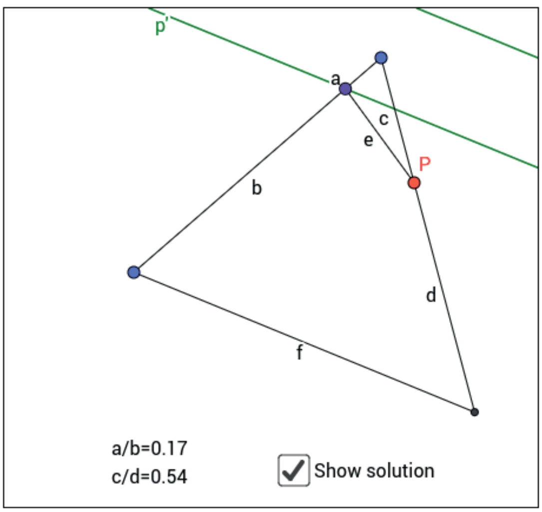
By focusing on inquiry learning, many other topics in elementary geometry can be discussed in the classroom via GG-ART. Following Artigue (2012) here we will focus only on such topics where the answer may be kept secret, hence the inquiry learning can be authentic.

In this part of the paper we focus on experiments that are of form: "What is the geometric locus of…?" From the algebraic geometry point of view, the answers here are limited to algebraic curves. Being in the classroom, the degree of the appearing curves should be mostly 1 or 2, unless the teacher wants to introduce higher order algebraic curves. This is, actually unavoidable when the students start to do free experiments by using the LocusEquation tool (see Kovács, 2016) for simple experiments on a triangle, in particular obtaining a lemniscate that is of degree 4).

The most straightforward way of locus experiments is to find converse statements of theorems from the curriculum, including:

1. (A converse of Pythagoras.) Given a triangle *ABC* with sides *a*, *b* and *c*. When *AB* is fixed, where to put *C* to have $a^2+b^2=c^2$?
2. (A converse of the right altitude theorem.) Given a triangle *ABC* with sides *a*, *b* and *c*. Let *p* and *q* be the orthogonal projections of *a* and *b* to *c*, respectively, and let *h* be the altitude with respect to *C*. When *AB* is fixed, where to put *C* to have $h^2=pq$?
3. (A converse of the intercept theorem.) A triangle with sides *a*+*b*, *c*+*d* and *f* is given as seen in Figure 3. A different split of side *a*+*b* is not allowed (that is, it is fixed), but for the side *c*+*d* the splitting point *P* can be freely chosen. Where to put *P* to have $a/b=c/d$?

FIGURE 3. *A converse of the intercept theorem*



Actually, in the two latter examples interesting extra components will appear in the resulting curves. In the second example the extra component is a hyperbola (the straightforward solution is the Thales circle of *AB*), see Abanades et al. (2016), while in the third one another line, parallel to *a*, appears (that is, the locus is a union of two parallel lines), see Kovács (2017).

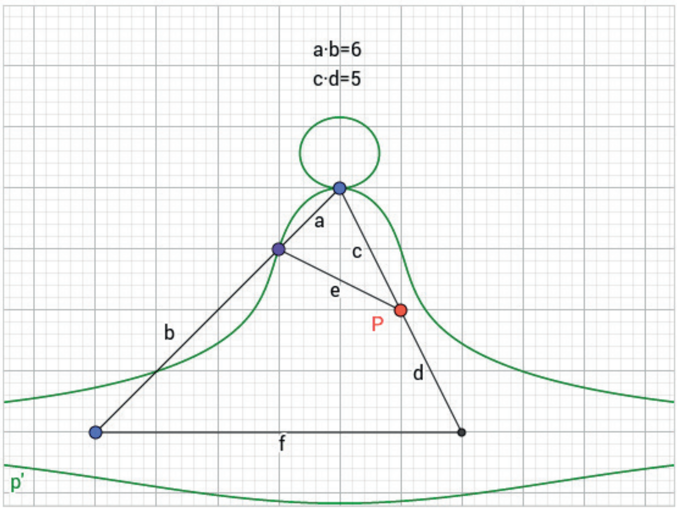
Two other examples contain simpler equations, but the related statements may have interesting generalizations:

4. (Triangle inequality and a definition of the ellipse.) Given a triangle *ABC* with sides *a*, *b* and *c*. Where to put *C* to have $a+b=c$? Or, to have $a+b=2c$? Or, to have $a+b=3c$? And how about $a+2b=c$? (This last one yields a quartic.)
5. (Perpendicular bisector and Apollonius' circle, that is the set of points whose distances from two fixed points are in a constant ratio.) Given a triangle *ABC*

with sides *a*, *b* and *c*. When *AB* is fixed, where to put *C* to have *a*=*b*? And how about *a*=2*b*?

Generalizations of the first three examples are also possible, and may result in unexpected curves. For instance, in the third example the question is somewhat artificial: why exactly *a*/*b*=*c*/*d* is to be investigated? When studying instead, say, the relation *ab*=*cd*, surprising geometric objects can be "discovered", but completely out of track of the curriculum (see Figure 4 and Kovács, 2017).

FIGURE 4. *A surprising locus*



Finally we mention two more tasks that can be used for inquiry learning and conveniently studied by GG-ART:

6. Let a fixed square *ABCD* be given, and also a free point *E*. Where to put *E* in order to have the triangle *ABE* the same area as of the square?
7. When are the diagonals of a parallelogram perpendicular?

Real-life examples

The National Science Education Standards (1996) outline six important aspects pivotal to inquiry learning in science education in the United States, including the extent to which students are able to learn with deep understanding will influence how transferable their new knowledge is to real life contexts.

Here we show two basic real life examples that can be used to link inquiry learning as real world connections. The first task is to consider a kite (Figure 5) and collect interesting facts about it. It can be observed, for example, that the kite has a symmetry axis that goes through on two opposite vertices. This fact implies that the pink quadrilateral that joins the midpoints of the sides of the kite, is actually

a rectangle. One can conclude that this figure is a special case of Varignon's theorem that states that the midpoint quadrilateral is a parallelogram.
The technical steps for the student when using GG-ART are to carefully construct the geometrical model of the kite by using circles to establish equal lengths, accordingly, then create the midpoints, join them, and conclude generally true properties by dragging the free points and using the Relation tool.

FIGURE 5. *A colored kite illustrating a special case of Varignon's theorem*

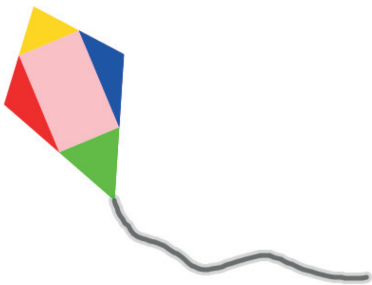
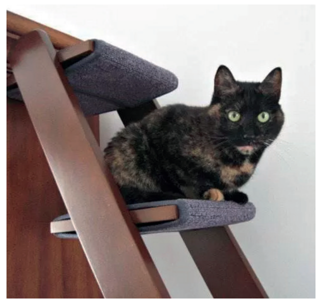
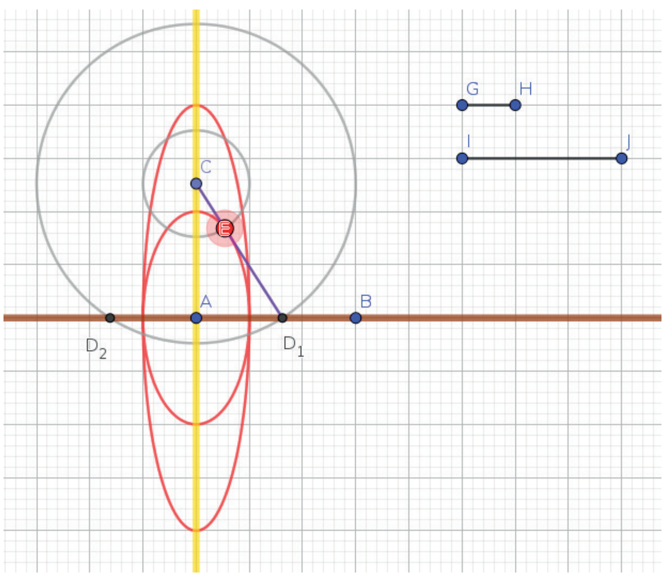


FIGURE 6. *Cat on a ladder (King, 2016)*



A second task we highlight is the locus of a falling cat that is sitting at a fixed position on a ladder which is sliding down wall (Figure 6). The task is about to determine the geometric movement of the cat. This example is actually a reformulation of the trammel of Archimedes, also known as an ellipsograph, that produces an elliptic motion (see Figure 7).

FIGURE 7. *A possible solution of the task "cat on a ladder" by using GG-ART*



A possible solution of the inquiry is as follows. The floor is defined as line *AB*, and the wall as a perpendicular line to it at point *A*. Let *C* be an arbitrary point of the wall: this will play the role of the top of the ladder. The length of the ladder is defined as the length of segment *IJ*. By drawing a circle with center *C* and radius *IJ*, the intersection point of the circle and the floor will designate the bottom of the ladder. The cat sits on the ladder and has a fixed distance *GH* from the top. Therefore another circle with center *C* and radius *GH* designates the position *E* of the cat as an intersection of the ladder and this second circle. Finally the command **LocusEquation(E,C)**, that is the first (explicit) form in GG-ART, computes and displays the motion curve of the cat, actually the fourth of an ellipse. The other three fourth of the inner ellipse and also the whole outer ellipse in Figure 7 do not play any role in the geometric solution of the task, they are just displayed as algebraic solutions that have no meaning in the real-life problem.

Let us highlight again that the solution of this problem consists of two ellipses. Discussing the properties or even the mathematical definition of an ellipse may be out of track of the curriculum in many countries. However, it is clear that many real-life problems involve the study of higher order curves, hence it seems to make sense to reconsider teaching conic sections at least at a basic level.

4. Conclusion

A potential approach for inquiry learning

In the previous sections we sketched up some possible inquiry tasks by using the dynamic geometry software tool GeoGebra and its novel automated reasoning tools. Despite the technology seems to be mature to support it, we admit that its classroom utilization is still in an experimental phase. By disseminating the availability of these novel tools we hope that automated reasoning could help students at various knowledge levels to start playing with their own experiments, and deepen their own mathematical thinking.

A relevant consideration here, regarding the potential impact of GG-ART in mathematics education, is the current availability of GeoGebra (GG), the program over which we have implemented our automatic reasoning tools (ART), over computers, tablets and smartphones, with and without internet connection, and backed up by a well spread community of millions of teachers and students, all over the world. Thus, our methodological proposal is not just an academic disquisition, but could have a *real effect* in the development of an inquiry based geometry education. In this direction, our final goal should be to launch an open call to the community of maths teachers and maths education researchers, to help improving the applicability of this powerful and novel instrument.

We have mentioned in the Introduction that our proposal differs from those related to automatic, intelligent tutoring in geometry, in that in our case the automatism is not on the tutoring protocol, but on the availability, for the student, of a tool with (roughly speaking) unlimited geometric knowledge. We think both approaches can be very useful as they are, in some sense, complementary, since GG-ART can be considered as a reliable augur, but the student needs as well some instructions—from an automated tutor—about what to ask to the prophet! Perhaps we should go one further step ahead and consider, in the future, the possibility of automatizing—by using GG-ART—the merging of both tools through the automatic creation of the tutoring devices… i.e. automatizing the elaboration of automated mentors! See Font et al. (2018) for a pioneer proposal in this direction.

Further work towards inquiry learning

In our on-going research, focusing on inquiry learning, another important aim is to study if

- using GG-ART as a learning activity makes inquiry learning for students *indeed* possible,
- and to *which extent* the criteria of inquiry learning are fulfilled.

A later step should be, based on the study, to improve GG-ART to be compliant with inquire learning to a high extent.

For the first viewpoint of the study we will focus on Ulm (2009, p. 90) that argues that inquiry learning takes place if learners at least partially get familiar with a topic area which was unknown to them and seemed complex before by means of independent cognitive activity.

The construct inquiry learning can be made quantifiable by use of a special post-interventional inventory called *Criteria of Learning Inventory* (CILI). The inventory, first published by Reitinger (2016), can be used as a standardized inventory to measure the evolvement of inquiry learning within educational learning settings in tertiary education (Reitinger, 2016, p. 55).

In particular we plan to study if the learning-setting with GG-ART allows inquiry learning according to the criteria of inquiry leaning to a concept of Reitinger's. The theoretical partial construct embodied in the inventory are *Experience-based Hypothesing*, *Authentic Exploration*, *Critical Discourse*, and *Conclusio-based Transfer*. These constructs are operation into twelve English-language items that refer to an experienced learning activity (Reitinger, 2016, p. 45).

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